

COMBINATION OF AFFINE DEFORMATIONS ON A HYPERBOLIC SURFACE

TAKAYUKI MASUDA

ABSTRACT. This paper is a continuation of the previous paper of the author[M]. We show that an affine deformation space of a hyperbolic surface of type (g, b) can be parametrized by Margulis invariants and affine twist parameters with a certain decomposition of the surface, which are associated with the Fenchel-Nielsen coordinates in Teichmüller theory. W.Goldman and G.Margulis[GM] introduced that a translation part of an affine deformation canonically corresponds to a tangent vector on the Teichmüller space. By this correspondence, we explicitly represent tangent vectors on the Teichmüller space from the perspective of Lorentzian geometry, only when the tangent vectors correspond to Fenchel-Nielsen twists along separating geodesic curves on a hyperbolic surface.

1. INTRODUCTION

This paper is a continuation of the previous paper of the author[M]. Let $G \subset \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}^o(2, 1)$ be a finitely generated Fuchsian group. We suppose that G has only hyperbolic elements and a quotient hyperbolic surface \mathbb{H}^2/G has at least one hole. A *cocycle* \mathbf{u} on G is a map from G to $(2+1)$ -dimensional Lorentzian spacetime \mathbb{R}_1^2 , which satisfies the cocycle condition. An affine deformation $\rho_{\mathbf{u}}$ of G is a homomorphism from G to $\mathrm{SO}^o(2, 1) \ltimes \mathbb{R}_1^2$, defined by $g \mapsto (g, \mathbf{u}(g))$.

Following fundamental works by [DG1][CDG1][CDG2][CDG3], we regard the first cohomology group $H^1(G, \mathbb{R}_1^2)$ as the affine deformation space of G . They classify all proper affine deformations of $H^1(G, \mathbb{R}_1^2)$ for some Fuchsian groups G . A *Margulis invariant* $\mathrm{Mar}_{\mathbf{u}}$ is, by definition, a map which sends each element of G to the translation length in \mathbb{R}_1^2 .

The author's previous work is related to any hyperbolic surface $S_{0,b}$ ($b \geq 4$) without cusps. By fixing the pants-decomposition of $S_{0,b}$, we show that the affine deformation space $H^1(G_{0,b}, \mathbb{R}_1^2)$ is parametrized by Margulis invariants (corresponding to the original boundary components and the dividing curves of the pants-decomposition) and the

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affine twist parameters (along the dividing curves). The aim of this paper is to discuss such kind of coordinates for arbitrary hyperbolic surface $S_{g,b}$ with non empty boundary.

1.1. Affine deformations of $G_{g,b}$. In the previous paper [M], the author introduced the *affine twist cocycle*. We put a reference point(cocycle) in $H^1(G_{g,b}, \mathbb{R}_1^2)$. For any cocycle \mathbf{u} on $G_{g,b}$, we can determine how much \mathbf{u} has the affine twist cocycles. We call the quantity an *affine twist parameter*.

Now we will parametrize $H^1(G_{g,b}, \mathbb{R}_1^2)$; we decompose $S_{g,b}$ into g handles and $(g + b - 2)$ pairs of pants. (See Figure 3.) We notice that this decomposition is associated with the Fenchel-Nielsen coordinates in Teichmüller theory.

Theorem 1.1. *The affine deformation space $H^1(G_{g,b}, \mathbb{R}_1^2)$ can be linearly parametrized by the Margulis invariants and the affine twist parameters with respect to the above decomposition under the assumption that each set of generators of the once-holed tori does not have an angle $\pi/2$. (Here the angles will be defined in §2.)*

1.2. Infinitesimal deformation of $S_{g,b}$. In [GM], Goldman and Margulis discovered a certain relation between Margulis invariants and infinitesimal deformations of hyperbolic structures by using the identification between \mathbb{R}_1^2 and the Lie algebra $sl_2(\mathbb{R})$ of $PSL(2, \mathbb{R})$. As was shown in [M], the affine twist cocycles for the special loops are recognized as infinitesimal deformations of Fenchel-Nielsen twist deformations of $S_{0,b}$. Indeed, our affine twist cocycle satisfies the cosine formula which is an analogous to Wolpert's formula for Fenchel-Nielsen twist (see [W]). Let $\ell : G_{g,b} \rightarrow \mathbb{R}$ be a (hyperbolic) translation length. In this paper, we extend this recognition as follows:

Theorem 1.2. *On a hyperbolic surface $S_{g,b}$, consider any geodesic loop $\sigma \in \pi_1(S)$. Suppose that another geodesic loop h separates the S into two surfaces whose interiors are disjoint. Let $\sigma(t) (t \in \mathbb{R})$ be a deformation of σ by \mathbf{AT}_h under the infinitesimal deformation of Goldman-Margulis. (Here $\sigma(0) = \sigma$.) Let $\ell_\sigma(t) := \ell(\sigma(t))$. Then a rate of the infinitesimal deformation of hyperbolic length $\ell(\sigma) = \ell_\sigma(0)$ is*

$$(1) \quad \left. \frac{d\ell_\sigma}{dt}(t) \right|_{t=0} = 2 \sum_{p \in h \cap \sigma} \cos(\theta_h^\sigma)_p,$$

where $(\theta_h^\sigma)_p$ is an angle at $p \in S_{g,b}$, which is defined in §2.

For non-separating loops on $S_{g,b}$, this result is still open.

This paper is organized as follows: Basic notations and definitions are introduced in §2. In order to consider the affine deformation space of $S_{g,b}$, we treat affine deformations of its handles. Namely, we parametrize the affine deformation space of a once-holed sphere by the Margulis invariants in §3. Then we prove Proposition 3.5. In next section §4, Theorem 1.1 is shown, and an important problem is raised. Finally, in §5, we calculate the correspondences of the infinitesimal deformations of Goldman-Margulis. Namely, we prove Theorem 1.2.

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2. SETTING

Here we introduce some basic notations and definitions.

2.1. Basic notations. Let $S_{g,b}$ be a hyperbolic surface homeomorphic to a compact orientable surface of genus g with b boundary components. We denote by $G_{g,b}$ the fundamental group of $S_{g,b}$, which is naturally considered as a Fuchsian group associated with $S_{g,b}$. We always identify a closed geodesic curve with an element of $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}^o(2, 1)$.

2.2. Lorentzian Geometry. A $(2+1)$ -Lorentzian spacetime \mathbb{R}_1^2 is an affine space whose associated inner product, called a *Lorentzian inner product*, is defined by $B([x_1, x_2, x_3], [y_1, y_2, y_3]) = x_1y_1 + x_2y_2 - x_3y_3$ over the canonical basis in \mathbb{R}_1^2 . A set of future-pointing rays in the interior of the upper part of the light cone (with respect to a certain reference point in \mathbb{R}_1^2) is regarded as a Klein-Poincare hyperbolic disk model \mathbb{H}^2 in \mathbb{R}_1^2 , which is induced from the inner product B (see [CDG1] for detail.).

The following definitions are introduced in [DG1, CDG1]. An affine isometry group of \mathbb{R}_1^2 is isomorphic to the twisted product $\mathrm{SO}^o(2, 1) \ltimes \mathbb{R}_1^2$. Every element η of this group is represented as a pair $(h, \mathbf{u}(h))$ for $h \in \mathrm{SO}^o(2, 1)$ and $\mathbf{u}(h) \in \mathbb{R}_1^2$. A *hyperbolic* element h has three distinct real eigenvalues. We choose three normalized eigenvectors as follows:

- (i) The future-pointing null vector \mathbf{X}_h^- has the smallest eigenvalue and the Euclidean norm is 1.
- (ii) The future-pointing null vector \mathbf{X}_h^+ has the largest one, and the Euclidean norm is 1.
- (iii) The unit spacelike vector \mathbf{X}_g^0 has 1 as an eigenvalue and its orientation is defined by $\det(\mathbf{X}_h^0, \mathbf{X}_h^-, \mathbf{X}_h^+) > 0$.

Note that the subspace $\langle \mathbf{X}_h^-, \mathbf{X}_h^+ \rangle_{\mathbb{R}}$ generated by \mathbf{X}_h^- and \mathbf{X}_h^+ coincides with the orthogonal complements $(\mathbf{X}_h^0)^\perp$ of \mathbf{X}_h^0 . A transformation η

is also called *hyperbolic* if the linear part h is hyperbolic. The set $\{\mathbf{X}_h^0, \mathbf{X}_h^-, \mathbf{X}_h^+\}$ is called a *null frame* of h (or η). The following lemma is well known.

Lemma 2.1. *Let h_1, h_2 be hyperbolic elements in $\mathrm{SO}^o(2, 1)$ whose unique oriented invariant axes in \mathbb{H}^2 intersect. The angle θ between tangent vectors of them at their intersection satisfies $B(\mathbf{X}_{h_1}^0, \mathbf{X}_{h_2}^0) = \cos \theta$.*

In a surface (resp. \mathbb{H}^2), let us denote by $(\theta_{h_1}^{h_2})_p$ an angle between two oriented geodesic loops (resp. unique invariant axes) h_1, h_2 at their intersection p . The choice of the angle is the one, seen h_2 (forget its orientation) in the left-hand direction along the direction of the orientation of the h_1 . Notice that the angle $(\theta_{h_1}^{h_2})_p \in (0, \pi) \subset \mathbb{R}$. We may omit the subscript for a point when the point is clear from context.

2.3. Affine deformations of a hyperbolic surface. A homomorphism $\rho : G_{g,b} \hookrightarrow \mathrm{SO}^o(2, 1) \ltimes \mathbb{R}_1^2$ is called an *affine deformation* if $\rho(h) = (h, \mathbf{u}(h))$ for all $h \in G_{g,b}$. The translation part is called a *cocycle* $\mathbf{u} : G_{g,b} \rightarrow \mathbb{R}_1^2$. The cocycle satisfies a *cocycle condition*: $\mathbf{u}(h_1 h_2) = h_1 \mathbf{u}(h_2) + \mathbf{u}(h_1)$ for $h_1, h_2 \in G_{g,b}$. A *coboundary* $\delta_{\mathbf{v}}$ is a cocycle which forms $\delta_{\mathbf{v}}(h) = \mathbf{v} - h\mathbf{v} \in (\mathbf{X}_h^0)^\perp$ for some $\mathbf{v} \in \mathbb{R}_1^2$. The coboundary $\delta_{\mathbf{v}}$ corresponds to a translation by \mathbf{v} . Denote a space of cocycles (resp. coboundaries) by $Z^1(G_{g,b}, \mathbb{R}_1^2)$ (resp. $B^1(G_{g,b}, \mathbb{R}_1^2)$). A quotient space $H^1(G_{g,b}, \mathbb{R}_1^2) := Z^1(G_{g,b}, \mathbb{R}_1^2)/B^1(G_{g,b}, \mathbb{R}_1^2) = \{[\mathbf{u}] \mid \mathbf{u} \in Z^1(G_{g,b}, \mathbb{R}_1^2)\}$ is regarded as the space of affine deformations of $G_{g,b}$.

2.4. Margulis invariant. If a hyperbolic element $\eta = (h, \mathbf{u}(h))$ acts freely on \mathbb{R}_1^2 , it admits a unique invariant axis C_η . On C_η , η acts as just a translation. The translation distance with respect to B is called the *Margulis invariant* $\mathrm{Mar}_{\mathbf{u}}(h)$. The Margulis invariant coincides with $B(\eta(x) - x, \mathbf{X}_h^0)$ for any $x \in \mathbb{R}_1^2$ (Refer to [Ma] for the details.). Then the translation part of η is represented as:

$$(2) \quad \mathbf{u}(h) = \mathrm{Mar}_{\mathbf{u}}(h) \mathbf{X}_h^0 + c^- \mathbf{X}_h^+ + c^+ \mathbf{X}_h^+,$$

for some real numbers c^\pm . One of the properties is:

Lemma 2.2 ([DG2, CD]). *Let $\mathbf{u}, \mathbf{u}' \in Z^1(G_{g,b}, \mathbb{R}_1^2)$. Assume that $\mathrm{Mar}_{\mathbf{u}}(h) = \mathrm{Mar}_{\mathbf{u}'}(h)$ for all $h \in G_{g,b}$. Then $[\mathbf{u}] = [\mathbf{u}']$ holds.*

3. STRUCTURES ON ONCE-HOLED TORUS

3.1. Hyperbolic geometry of once-holed torus. Let $S_{1,1}$ be a hyperbolic surface homeomorphic to a once-holed torus. The fundamental group $G_{1,1}$ is isomorphic to a free group $\langle w_1, w_2 \rangle$ of rank two, where w_1 and w_2 are simple closed curves corresponding to a longitude loop and

a meridian loop in $S_{1,1}$ respectively (see Figure 1). The actions by the generators w_1, w_2 are illustrated in Figure 2.

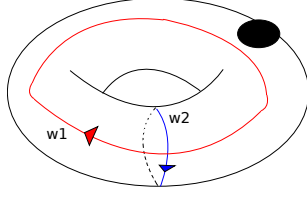


FIGURE 1.

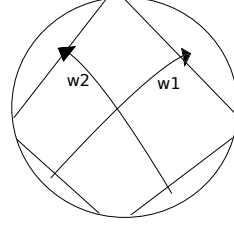


FIGURE 2.

When we cut $S_{1,1}$ along the loop w_2 , we can get a pair of pants with g_1, g_2, g_3 as the boundary components;

$$(3) \quad g_1 := [w_1, w_2], \quad g_2 := w_2, \quad g_3 := w_1 w_2^{-1} w_1^{-1},$$

where $[w_1, w_2]$ is a commutator of w_1 and w_2 . On $S_{1,1}$, the loop g_1 equals to the unique boundary component and g_2, g_3 are same loop. Let P denote a group generated by g_1, g_2 and g_3 . Note that $w_1 \notin P$. We set their null frames as follows:

$$\begin{aligned} w_1 &\leftrightarrow \{\mathbf{Y}_1^0, \mathbf{Y}_1^-, \mathbf{Y}_1^+\}, \\ g_1 &\leftrightarrow \{\mathbf{X}_1^0, \mathbf{X}_1^-, \mathbf{X}_1^+\}, \\ g_2 = w_2 &\leftrightarrow \{\mathbf{X}_2^0, \mathbf{X}_2^-, \mathbf{X}_2^+\} = \{\mathbf{Y}_2^0, \mathbf{Y}_2^-, \mathbf{Y}_2^+\}, \\ g_3 &\leftrightarrow \{\mathbf{X}_3^0, \mathbf{X}_3^-, \mathbf{X}_3^+\}. \end{aligned}$$

3.2. Affine deformations of $G_{1,1}$. The purpose of this part is to represent cocycles on $G_{1,1}$ by using the Margulis invariant of g_1 . At first we check an arbitrary property of Margulis invariant of g_1 .

Lemma 3.1 (existence). *Assume $\theta_{w_1}^{w_2} \neq \pi/2$. For any $\zeta_1, \zeta_2, \kappa \in \mathbb{R}$, there exists a cocycle \mathbf{u} on $G_{1,1}$ such that, for some real numbers $d_1^\pm, d_2^\pm, c_3^\pm \in \mathbb{R}$,*

$$\begin{aligned} \mathbf{u}(w_1) &= \zeta_1 \mathbf{Y}_1^0 + d_1^- \mathbf{Y}_1^- + d_1^+ \mathbf{Y}_1^+, \\ \mathbf{u}(w_2) &= \zeta_2 \mathbf{Y}_2^0 + d_2^- \mathbf{Y}_2^- + d_2^+ \mathbf{Y}_2^+, \\ \mathbf{u}(g_1) &= \kappa \mathbf{X}_1^0 + c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+. \end{aligned}$$

If $\theta_{w_1}^{w_2} = \pi/2$, an equation

$$(4) \quad \frac{\kappa}{K_{\frac{\pi}{2}}} = (1 + \lambda_1)(-1 + \lambda_2)\zeta_1 + (-1 + \lambda_1)(1 + \lambda_2)\zeta_2$$

must be satisfied, where

$$(5) \quad K_{\frac{\pi}{2}} := \frac{-2}{\sqrt{(-1 + \lambda_2)^2 + \lambda_1^2(-1 + \lambda_2)^2 - 2\lambda_1(1 + 6\lambda_2 + \lambda_2^2)}}.$$

Here the $\lambda_1, \lambda_2 (> 0)$ are the smallest eigenvalues of w_1 and w_2 respectively.

Proof. By the cocycle condition and a direct calculation,

$$(6) \quad \mathbf{u}(g_1) = (Id - g_3^{-1})\mathbf{u}(w_1) + (w_1 - g_1)\mathbf{u}(w_2)$$

holds for any cocycle \mathbf{u} on the $S_{1,1}$.

If $\mathbf{u}(w_1) = \zeta_1 \mathbf{Y}_1^0 + a \mathbf{Y}_1^- + b \mathbf{Y}_1^+$, $\mathbf{u}(w_2) = \zeta_2 \mathbf{Y}_2^0 + c \mathbf{Y}_2^- + d \mathbf{Y}_2^+$, then we obtain a representation of $\mathbf{u}(g_1)$ by (6). An inner product with \mathbf{X}_1^0 produces a Margulis invariant of g_1 .

$$\begin{aligned} \text{Mar}_{\mathbf{u}}(g_1) &= \zeta_1 B((Id - g_3^{-1})\mathbf{Y}_1^0, \mathbf{X}_1^0) + \zeta_2 B((w_1 - g_1)\mathbf{Y}_2^0, \mathbf{X}_1^0) \\ &\quad + a B((Id - g_3^{-1})\mathbf{Y}_1^-, \mathbf{X}_1^0) + b B((Id - g_3^{-1})\mathbf{Y}_1^+, \mathbf{X}_1^0) \\ &\quad + c B((w_1 - g_1)\mathbf{Y}_2^-, \mathbf{X}_1^0) + d B((w_1 - g_1)\mathbf{Y}_2^+, \mathbf{X}_1^0). \end{aligned}$$

Claim. The following hold:

- (i) $\text{Mar}_{\mathbf{u}}(g_1)$ does not depend on a, b, c, d at all if $\theta_{w_1}^{w_2} = \pi/2$, namely the coefficients of a, b, c, d are zero,
- (ii) $\text{Mar}_{\mathbf{u}}(g_1)$ does linearly on $a, b, c, d, \zeta_1, \zeta_2$ (otherwise).

Proof. By conjugation, we may set $\mathbf{Y}_1^0 := [1, 0, 0]$, $\mathbf{Y}_1^\pm := \frac{1}{\sqrt{2}}[0, \mp 1, 1]$, and also $\mathbf{Y}_2^\square := [\theta\text{-rotation around } z\text{-axis of } \mathbf{Y}_1^\square]$ where $\theta \in (0, \pi)$, and $\square \in \{0, +, -\}$. Their matrices can be represented as

$$w_1 = [\mathbf{Y}_1^0, \mathbf{Y}_1^-, \mathbf{Y}_1^+] \begin{bmatrix} 1 & & \\ & \lambda_1 & \\ & & \lambda_1^{-1} \end{bmatrix}, w_2 = [\mathbf{Y}_2^0, \mathbf{Y}_2^-, \mathbf{Y}_2^+] \begin{bmatrix} 1 & & \\ & \lambda_2 & \\ & & \lambda_2^{-1} \end{bmatrix}.$$

We can find a constant $K (\neq 0) \in \mathbb{R}$ which depends only on the hyperbolic structure of $S_{1,1}$ and satisfies:

$$(7) \quad \begin{aligned} \frac{\text{Mar}_{\mathbf{u}}(g_1)}{K} &= \sin \theta \{ (1 + \lambda_1)(-1 + \lambda_2)\zeta_1 + (1 + \lambda_2)(-1 + \lambda_1)\zeta_2 \} \\ &\quad - \sqrt{2} \cos \theta \{ (-1 + \lambda_2)(a - \lambda_1 b) - (-1 + \lambda_1)(c - \lambda_2 d) \}. \end{aligned}$$

Note that the constant K is not zero regardless of hyperbolic structures of the once-holed torus. Thus we can check our claim. \square

In order to prove the lemma, we have only to find a solution (a, b, c, d) of an equation $\kappa = \text{Mar}_{\mathbf{u}}(g_1)$ for the given $(\zeta_1, \zeta_2, \kappa) \in \mathbb{R}^3$. However it is easily checked. \square

From now on, we suppose that $\theta_{w_1}^{w_2} \neq \pi/2$. In order to show Proposition 3.5, we consider a normalization of the representation of cocycles up to translation. At first, we prove two lemmas.

Lemma 3.2. *For the cocycle \mathbf{u} in Lemma 3.1, there exists the unique vector $\mathbf{v} \in \mathbb{R}_1^2$ such that*

$$\begin{aligned} (\mathbf{u} + \delta_{\mathbf{v}})(w_1) &= \zeta_1 \mathbf{Y}_1^0 + \mathbf{f}_1^- + \mathbf{f}_1^+, \\ (\mathbf{u} + \delta_{\mathbf{v}})(w_2) &= \zeta_2 \mathbf{Y}_2^0 + \mathbf{f}_2^- + \mathbf{0}, \\ (\mathbf{u} + \delta_{\mathbf{v}})(g_1) &= \kappa \mathbf{X}_1^0 + \mathbf{0} + \mathbf{0}, \end{aligned}$$

where $\mathbf{f}_1^\pm \in \mathbb{R}\mathbf{Y}_1^\pm$, $\mathbf{f}_2^- \in \mathbb{R}\mathbf{Y}_2^-$ and they depend on $(\zeta_1, \zeta_2, \kappa, d_1^-, d_1^+, d_2^-, d_2^+) \in \mathbb{R}^6$.

Proof. Explicitly, we set

$$\mathbf{v} := s\mathbf{X}_1^0 + \frac{(c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+) - g_1^{-1}(c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+)}{(1 - \mu_1)(1 - \mu_1^{-1})},$$

where $(0 <) \mu_1 < \mu^{-1}$ are eigenvalues of g_1 and s is the suitable real number. We will show that this vector satisfies this lemma. It is easily checked that $(\mathbf{u} + \delta_{\mathbf{v}})(g_1) = \kappa \mathbf{X}_1^0$.

We have only to show that $\delta_{\mathbf{X}_1^0}(w_2)$ has the direction of \mathbf{Y}_2^+ . Then, for the suitable number s , $\delta_{\mathbf{v}}(w_2)$ deletes the direction of \mathbf{Y}_2^+ in the representation of the cocycle of w_2 . In order to find the direction of \mathbf{Y}_2^+ in $\delta_{\mathbf{X}_1^0}(w_2)$, we show that $B(\delta_{\mathbf{X}_1^0}(w_2), \mathbf{Y}_2^-) \neq 0$ as follows:

$$\begin{aligned} B(\delta_{\mathbf{X}_1^0}(w_2), \mathbf{Y}_2^-) &= B(\mathbf{X}_1^0 - w_2 \mathbf{X}_1^0, \mathbf{Y}_2^-) \\ &= (1 - \lambda_2^{-1})B(\mathbf{X}_1^0, \mathbf{Y}_2^-). \end{aligned}$$

Note that “ $1 - \lambda_2^{-1}$ ” is not zero. Since \mathbf{X}_1^0 is the principal eigenvector of $w_1 w_2 w_1^{-1} w_2^{-1}$, the null vectors \mathbf{X}_1^+ and \mathbf{X}_1^- are near \mathbf{Y}_1^+ and \mathbf{Y}_2^+ respectively. So the null vector \mathbf{Y}_2^- is not contained the orthogonal plane of \mathbf{X}_1^0 . Thus $B(\mathbf{X}_1^0, \mathbf{Y}_2^-) \neq 0$ holds. Therefore, $B(\delta_{\mathbf{X}_1^0}(w_2), \mathbf{Y}_2^-) \neq 0$ as desired. \square

Lemma 3.3. *For the cocycle in Lemma 3.2, the vectors $\mathbf{f}_1^\pm, \mathbf{f}_2^-$ do not depend on c_1^\pm, d_1^\pm and d_2^\pm for the fixed triple $(\zeta_1, \zeta_2, \kappa)$.*

Proof. From Lemma 6.1 in [CDG1], a triple $(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ with $\zeta_3 := \text{Mar}_{\mathbf{u}}(w_1 w_2)$ determines a cohomology class of \mathbf{u} . Under the condition

of the conjugation as in Lemma 3.1, the following also holds:

(8)

$$\begin{aligned} \frac{\zeta_3}{K'} &= -\{(1 + \lambda_1)(-1 + \lambda_2) \cot \theta + (-1 + \lambda_1)(1 + \lambda_2) \csc \theta\} \zeta_1 \\ &\quad - \{(-1 + \lambda_1)(1 + \lambda_2) \cot \theta + (1 + \lambda_1)(-1 + \lambda_2) \csc \theta\} \zeta_2 \\ &\quad - \sqrt{2}\{(-1 + \lambda_2)(a - \lambda_1 b) - (-1 + \lambda_1)(c - \lambda_2 d)\}, \end{aligned}$$

where the constant K' also depends only on the hyperbolic structure on $S_{1,1}$, and $K' \neq 0$ even if $\theta_{w_1}^{w_2} = \pi/2$. When two equations (7) and (8) are combined to eliminate the four terms a, b, c, d , we can obtain an equation between κ and ζ_3 obviously with ζ_1, ζ_2 fixed:

$$(9) \quad \frac{\kappa}{K} - \cos \theta \cdot \frac{\zeta_3}{K'} = \frac{2(\lambda_1 \lambda_2 - 1)}{\sin \theta} \cdot (\zeta_1 + \zeta_2).$$

Therefore κ determines ζ_3 and vice versa. Thus the triple $(\zeta_1, \zeta_2, \kappa)$ determines a cohomology class of \mathbf{u} .

Consider the cocycle in Lemma 3.2. The vectors $\mathbf{f}_1^\pm, \mathbf{f}_1^-$ are noticed not to depend on c_1^\pm, d_1^\pm and d_2^\pm , since its cohomology class is already determined. \square

Here we denote the cocycle in Lemma 3.3 by \mathbf{u}_T . Namely, \mathbf{u}_T can be written by

$$\begin{aligned} \mathbf{u}_T(w_1) &:= \zeta_1 \mathbf{Y}_1^0 + \mathbf{f}_1^-(\zeta_1, \zeta_2, \kappa) + \mathbf{f}_1^+(\zeta_1, \zeta_2, \kappa), \\ \mathbf{u}_T(w_2) &:= \zeta_2 \mathbf{Y}_2^0 + \mathbf{f}_2^-(\zeta_1, \zeta_2, \kappa) + \mathbf{0}, \\ \mathbf{u}_T(g_1) &:= \kappa \mathbf{X}_1^0 + \mathbf{0} + \mathbf{0}, \end{aligned}$$

where $\mathbf{f}_1^\pm(\zeta_1, \zeta_2, \kappa) \in \mathbb{R} \mathbf{Y}_1^\pm$ and $\mathbf{f}_2^-(\zeta_1, \zeta_2, \kappa) \in \mathbb{R} \mathbf{Y}_2^-$.

Definition 3.4. We define a map Φ_T by

$$\mathbb{R}^3 \ni (\zeta_1, \zeta_2, \kappa) \mapsto \mathbf{u}_T \in Z^1(G_{1,1}, \mathbb{R}_1^2).$$

Note that the Φ_t is a linear map. Namely, $\mathbf{f}_1^\pm, \mathbf{f}_2^- : \mathbb{R}^3 \rightarrow \mathbb{R}_1^2$ are linear.

Proposition 3.5. For every triple $(\zeta_1, \zeta_2, \kappa) \in \mathbb{R}^3$, there exists a cocycle \mathbf{u} on $G_{1,1}$ such that $\text{Mar}_{\mathbf{u}}(g_1) = \zeta_1, \text{Mar}_{\mathbf{u}}(w_2) = \zeta_2, \text{Mar}_{\mathbf{u}}(w_1) = \kappa$. Furthermore this correspondence is extended to a linear isomorphism from \mathbb{R}^3 to $H^1(G_{1,1}, \mathbb{R}_1^2)$. This result, however, does not hold in the case where $\theta_{w_1}^{w_2} \neq \frac{\pi}{2}$.

Proof. The map Φ_T naturally induces a well-defined map

$$\overline{\Phi_T} : \mathbb{R}^3 \ni (\zeta_1, \zeta_2, \kappa) \mapsto [\Phi_T(\zeta_1, \zeta_2, \kappa)] \in H^1(G_{1,1}, \mathbb{R}_1^2).$$

One can easily check that the map is a linear isomorphism; Indeed, the injectivity of $\overline{\Phi_T}$ follows from Lemma 2.2. The surjectivity follows from the equation (9) in Lemma 3.3, since it indicates the equivalence between κ and ζ_3 . \square

This Proposition 3.5 is for the special generator w_1 and w_2 . In fact, we will use this statement to prove Theorem 1.1. However, the same discussion in the proof of Proposition 3.5 shows:

Corollary 3.6. *For any generators ω_1, ω_2 of $G_{1,1}$, a triple*

$$(\text{Mar}_{\mathbf{u}}(\omega_1), \text{Mar}_{\mathbf{u}}(\omega_2), \text{Mar}_{\mathbf{u}}([\omega_1, \omega_2])) \in \mathbb{R}^3$$

canonically determines a unique cocycle up to translation, if $\theta_{\omega_1}^{\omega_2} \neq \pi/2$.

Proof. We consider ω_1, ω_2 as elements of $\text{PSL}(2, \mathbb{R})$. Since they are the generators of $G_{1,1}$, their invariant axes in \mathbb{H}^2 intersects. Furthermore the commutator $[\omega_1, \omega_2]$ equals to $g_1^{\pm 1}$. Note that $\text{Mar}_{\mathbf{u}}(g_1) = \text{Mar}_{\mathbf{u}}(g_1^{-1})$. Therefore, this is the same condition with w_1 and w_2 up to conjugation. So we can calculate the Margulis invariant of g_1 (or g_1^{-1}) similarly. \square

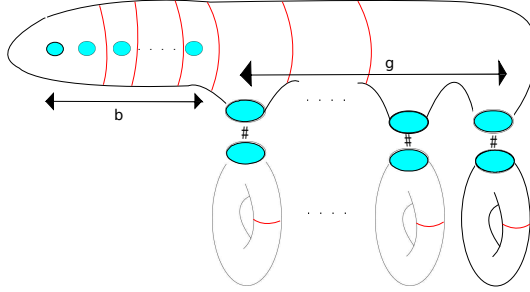
Remark 3.7. *V.Charette, T.Drumm, and W.Goldman ([CDG3]) describe the classification of affine deformations of $G_{1,1}$. Their discussion is due to the sets of generators of $G_{1,1}$. By considering the Margulis invariants of all primitive elements in $G_{1,1}$ (that is, there exists no element in $G_{1,1}$ such that these two elements generate $G_{1,1}$), they obtain the classification of proper affine deformations of $H^1(G_{1,1}, \mathbb{R}_1^2)$. In this paper, however, evaluating the Margulis invariant of the unique boundary component is needed, since we need to glue boundaries. Note that it is not a primitive element in $G_{1,1}$. An equation for the Margulis invariant for this element is introduced in [CG].*

4. COORDINATES OF AFFINE DEFORMATIONS FOR A HYPERBOLIC SURFACE WITH TYPE (g, b)

A goal of this section is to prove Theorem 1.1. Let $G_{g,b}$ be a Fuchsian group of the hyperbolic surface $S_{g,b}$. Assume that $S_{g,b}$ has at least one hole ($b > 0$) and no cusps. We will parametrize the affine deformation space $H^1(G_{g,b}, \mathbb{R}_1^2)$.

4.1. Decomposition. It is known that the dimension of $H^1(G_{g,b}, \mathbb{R}_1^2)$ is equal to $6g + 3b - 6$.

We take a certain decomposition of $S_{g,b}$ into $(g + b - 2)$ pairs of pants and g once-holed tori. Figure 3 illustrates this decomposition. We denote the components by $P_1, P_2, \dots, P_{g+b-2}$ (the pairs of pants) and T_1, \dots, T_g (the once-holed tori) such that

FIGURE 3. The decomposition of $S_{g,b}$.

- Let P_1, \dots, P_{b-1} contain the original holes of $S_{g,b}$ respectively.
- Furthermore P_{b+j} is originally attached by T_j , $0 \leq j \leq g-2$.
- Let P_i and P_{i+1} share a unique simple closed curve on $S_{g,b}$, $1 \leq i \leq g+b-3$.
- Let T_g attach P_{b+g-2} on $S_{g,b}$.

Set $\mathbf{I} := \{1, \dots, b-1\}$, $\mathbf{J} := \{1, \dots, g\}$ and $\mathbf{K} := \{1, \dots, b+g-3\}$.

We label principal loops in these surfaces as follows:

- The original boundaries : $\gamma^1, \dots, \gamma^b$ ($\gamma^i \in \pi_1(P_i), i \in \mathbf{I}$)
- The unique holes of T_j : g^1, \dots, g^b ($g^j \in \pi_1(T_j \cap P_{b+j}), j \in \mathbf{J}$)
- The boundaries of the pairs of pants except the above two kinds of loops : f_1, \dots, f_{b+g-3} , ($f_k \in P_k \cap P_{k+1}, k \in \mathbf{K}$)
- The longitude loop on T_j : $w_1^j, (j \in \mathbf{J})$
- The meridian loop on T_j : $w_2^j, (j \in \mathbf{J})$

We notice that the hyperbolic lengths of these loops and the twisting parameters achieve the Fenchel-Nielsen coordinates of $S_{g,b}$ in Teichmüller space on the surface of type (g, b) . The dimension are also $6g + 3b - 6$.

4.2. Combination. This part is due to the result in [M]. About the case of the surface $S_{g,b} - \sum_{j \in \mathbf{J}} T_j$ (holed sphere), it is shown that all cocycles are linearly parametrized (up to translation) by Margulis invariants of γ^i, g^j, f_k ($i \in \mathbf{I}, j \in \mathbf{J}, k \in \mathbf{K}$) and *affine twist parameters* of f_k ($k \in \mathbf{K}$). Here a discussion is to define *affine twist cocycle* and an associated parameter, *affine twist parameter* on $S_{g,b}$.

First of all, consider a fourth-holed sphere S_4 with fixed hyperbolic structure. The fundamental group G_4 is a free group of rank three, which is represented by

$$\langle g_1, g_2, g_3, g_4 | g_1 g_2 g_3 g_4 = id \rangle.$$

Let $Q_1 := \langle g_1, g_2 \rangle$ and $Q_2 := \langle g_3, g_4 \rangle$ be subgroups in G_4 . Set $f := g_2^{-1}g_1^{-1}$ (note that $f^{-1} = g_4^{-1}g_3^{-1}$), which corresponds to a dividing curve in S_4 . Take a cocycle $\mathbf{u}^\xi \in Z^1(Q_\xi, \mathbb{R}_1^2)$, $(\xi = 1, 2)$, and assume $\text{Mar}_{\mathbf{u}^1}(f) = \text{Mar}_{\mathbf{u}^2}(f)$.

Definition 4.1 (Combination). *A cocycle $\mathbf{u}^1 \#_f \mathbf{u}^2$ on G_4 can be defined as follows:*

$$\mathbf{u}^1 \#_f \mathbf{u}^2(h) = \begin{cases} \mathbf{u}^1(h) & (h \in Q_1), \\ \mathbf{u}^2(h) + \delta_{\mathbf{trans}}(h) & (h \in Q_2), \\ \text{by cocycle condition} & (\text{otherwise}), \end{cases}$$

where a (fixed) vector \mathbf{trans} is chosen to satisfy

$$(10) \quad \mathbf{u}^2(f) + \delta_{\mathbf{trans}}(f) = \mathbf{u}^1(f).$$

We realize that this equation leave one dimensional ambiguity of choice of $\mathbf{trans} \in \mathbb{R}_1^2$. Therefore, considering this direction achieves an concept of *affine twist cocycle*.

Definition 4.2 (Affine twist). *A cocycle \mathbf{AT}_f on G_4 defined as the following manner was named an affine twist cocycle along f .*

$$\mathbf{AT}_f(h) = \begin{cases} \mathbf{0} & (h \in Q_1), \\ \mathbf{X}_f^0 - h\mathbf{X}_f^0 & (h \in Q_2), \\ \text{by cocycle condition} & (\text{otherwise}). \end{cases}$$

Lemma 4.3 ([M]). *Cocycles $\mathbf{u}^1 \# \mathbf{u}^2 + \tau \mathbf{AT}_f$ generate $H^1(G_4, \mathbb{R}^4)$, where $\tau \in \mathbb{R}$, and $\mathbf{u}^1, \mathbf{u}^2$ satisfy the assumption in Definition 4.1, with a cohomology class of $\mathbf{u}^1, \mathbf{u}^2$ fixed.*

This lemma induces the following claim under a certain normalization.

Proposition 4.4. *For any $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta, \tau) \in \mathbb{R}^6$, there exists a unique cocycle \mathbf{u} up to translation such that*

$$\begin{aligned} \text{Mar}_{\mathbf{u}}(g_i) &= \alpha_i (i = 1, 2, 3, 4) \\ \text{Mar}_{\mathbf{u}}(f) &= \beta \\ \text{AfT}_{\mathbf{u}}(f) &= \tau, \end{aligned}$$

where AfT denotes a coefficient of \mathbf{AT}_f in \mathbf{u} . (to consider this coefficient, we need a normalization.)

The cases of $S_{0,b}$ ($b \geq 4$) are discussed similarly.

4.3. Proof of Theorem 1.1. Consider a hyperbolic surface $S_{1,b}$. Let g_1 be a simple closed curve on $S_{1,b}$, which is divided by g_1 into $S_1 := S_{0,b+1}$ and $S_2 := S_{1,1}$. We use the notation in §3, that is, let $G_{1,1} = \langle w_1, w_2 \rangle$ be a fundamental group with $S_2 = \mathbb{H}^2/G_{1,1}$.

Then we can consider *combinations* of cocycles and *affine twists* on $S_{1,b}$ without the case $\theta_{w_1}^{w_2} = \pi/2$.

Definition 4.5 (Combination). *Let \mathbf{u}^ξ be a cocycle on S_ξ , $\xi = 1, 2$. Assume that $\text{Mar}_{\mathbf{u}^1}(g_1) = \text{Mar}_{\mathbf{u}^2}(g_1)$. A cocycle $\mathbf{u}^1 \# \mathbf{u}^2$ on $G_{1,b}$ is defined similarly as Definition 4.1.*

Definition 4.6 (Affine twist). *A cocycle \mathbf{AT}_{g_1} on $G_{1,b}$ is defined similarly as Definition 4.2.*

Here we mention a remark about the translation “**trans**”.

Remark 4.7. *Suppose that $\mathbf{u}(g_1) = \alpha_1 \mathbf{X}_1^0 + c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+$ are already determined. Let μ_1 be the smallest eigenvalue of g_1 . We choose the vector*

$$\mathbf{trans} := \frac{(c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+) - g_1^{-1}(c_1^- \mathbf{X}_1^- + c_1^+ \mathbf{X}_1^+)}{(1 - \mu_1)(1 - \mu_1^{-1})}.$$

Then this vector meets the equation (10).

For cocycles $\mathbf{u}^1 \#_{g_1} \mathbf{u}^2 + \tau \mathbf{AT}_{g_1}$ ($\tau \in \mathbb{R}$) under the assumption in definition 4.5 and 4.6, we have a bijection.

$$\begin{aligned} H^1(\pi_1(S_1), \mathbb{R}_1^2) \times H^1(\pi_1(S_2), \mathbb{R}_1^2) \times \mathbb{R} &\rightarrow H^1(\pi_1(S), \mathbb{R}_1^2) \\ ([\mathbf{u}^1], [\mathbf{u}^2], \tau) &\mapsto [\mathbf{u}^1 \#_{g_1} \mathbf{u}^2 + \tau \mathbf{AT}_{g_1}] \end{aligned}$$

In this map, we choose the representative elements \mathbf{u}^ξ of $[\mathbf{u}^\xi]$ ($\xi = 1, 2$) respectively.

Proof of Theorem 1.1. We find a cocycle such that

$$\begin{aligned} \text{Mar}_{\mathbf{u}}(\gamma^i) &= \alpha^i, \text{Mar}_{\mathbf{u}}(g^j) = \kappa^j, \text{Mar}_{\mathbf{u}}(f_k) = \beta_k, \\ \text{Mar}_{\mathbf{u}}(w_1^j) &= \zeta_1^j, \text{Mar}_{\mathbf{u}}(w_2^j) = \zeta_2^j, \\ \text{AfT}_{\mathbf{u}}(g^j) &= \tau^j, \text{AfT}_{\mathbf{u}}(f_k) = \epsilon_k \end{aligned}$$

where $i \in \mathbf{I}, j \in \mathbf{J}, k \in \mathbf{K}$.

We cut $S_{g,b}$ cut into $S_{0,b+g} \cup \sum_{j \in \mathbf{J}} T^j$ like as Figure 3.

- (i) On $S_{0,b+g}$, following [M], we define a cocycle \mathbf{u}^0 linearly by using $(\alpha^i, \kappa^j, \beta_k, \epsilon_k), i \in \mathbf{I}, j \in \mathbf{J}, k \in \mathbf{K}$.
- (ii) On T^j , we define a cocycle \mathbf{u}^j linearly by using $(\kappa^j, \zeta_1^j, \zeta_2^j), j \in \mathbf{J}$. A method is, for example, to use the representation of Proposition 3.5.

(iii) Finally we define a cocycle \mathbf{u} depended linearly on all values inductively. We set

$$\mathbf{w}_1 := \mathbf{u}^0 \#_{g^1} \mathbf{u}^1 + \tau^1 \mathbf{A} \mathbf{T}_{g^1}$$

For $j \in \mathbf{J}$, we define $\mathbf{w}_j := \mathbf{w}_{j-1} \#_{g^j} \mathbf{u}^j + \tau^j \mathbf{A} \mathbf{T}_{g^j}$. A desired cocycle \mathbf{u} is just \mathbf{w}_g .

It is trivial that this construction gives a canonical linear isomorphism between $\mathbb{R}^{6g+3b-6}$ and $H^1(G_{g,b}, \mathbb{R}_1^2)$ under translation equivalence. \square

Remark 4.8. *We can parametrize the affine deformation space $H^1(G_{g,b}, \mathbb{R}_1^2)$ even if a handle T^j of $S_{g,b}$ satisfies $\theta_{w_1^j}^{w_2^j} = \pi/2$. We take a set of generator ω_1 and ω_2 of $\pi_1(T^j)$ with $\theta_{\omega_1}^{\omega_2} \neq \pi/2$. By Corollary 3.6, we have only to consider ω_1 and ω_2 instead of w_1 and w_2 .*

4.4. Gluing Margulis spacetimes. We define a gluing of surfaces. Let S_1, S_2 be hyperbolic surfaces with holes but no cusps. Suppose that S_1 and S_2 have a boundary component which is same length. Then let $S_1 \# S_2$ be a new hyperbolic surface glued along the boundary with an appropriate twisting parameter. An affirmative answer to the following problem provides us the properness of affine deformations.

Problem 4.9. *Let $M(S)$ and $M(R)$ be Margulis spacetimes, whose underlying hyperbolic surfaces are S and R respectively. Suppose that these hyperbolic surfaces have same boundary component ∂ . Let $S \#_{\partial} R$ be a new hyperbolic surface, which are glued by S and R along ∂ . Then $M(S) \#_{\tau \mathbf{A} \mathbf{T}_{\partial}} M(R)$ is a Margulis spacetimes for some $\tau = \text{AfT}_{\mathbf{u}}(\partial) \in \mathbb{R}$, which has $S \#_{\partial} R$ as its underlying hyperbolic space.*

5. DEFORMATION OF HYPERBOLIC STRUCTURES ALONG AFFINE TWIST COCYCLE

In this section, we show Theorem 1.2.

Proof. Let $S = S_1 \cup S_2$ be a hyperbolic surface where S_1 and S_2 are glued along a loop h . Let their fundamental groups denoted by Σ_1, Σ_2 respectively. From the lemma of Goldman-Margulis [GM], we have

$$\left. \frac{d\ell_{\sigma}}{dt}(t) \right|_{t=0} = 2 \text{Mar}_{\mathbf{A} \mathbf{T}_h}(\sigma),$$

where $\ell_{\sigma}(t) := \ell(\sigma(t))$, and $\sigma(t)$ is the deformation of the hyperbolic structure of σ with respect to $\mathbf{A} \mathbf{T}_h$. Thus we have only to calculate the Margulis invariants for the affine twist cocycle in order to prove

Theorem 1.2. We easily notice that every $\sigma \in \Sigma_\xi$ satisfy $\text{Mar}_{\mathbf{AT}_h}(\sigma) = 0$ ($\xi = 1, 2$).

In [M], the author observes that, for the simplest loop on S which passes both S_1 and S_2 , the equation (1) is shown to hold. Indeed, the following lemma is proved by the same idea. However, for completeness, we shall give a proof.

Lemma 5.1. *Take any loop σ, σ' in $\pi_1(S)$,*

$$\sigma = \sigma_1^1 \sigma_1^2 \sigma_2^1 \sigma_2^2 \cdots \sigma_n^1 \sigma_n^2, \quad \sigma' = \sigma_1^1 \sigma_1^2 \sigma_2^1 \sigma_2^2 \cdots \sigma_n^1,$$

where the loop σ_k^ξ is in Σ_ξ ($1 \leq k \leq n$). Then an equation

$$\text{Mar}_{\mathbf{AT}_h}(\sigma) = \sum_{j=1}^n \cos(\theta_h^\sigma)_{p_j} + \sum_{j=1}^n \cos(\theta_h^\sigma)_{q_j}$$

holds, where points p_j, q_j are intersections of σ, h on S such that a tangent vector at p_j along h orients a side of S_2 , and q_j is otherwise.

Proof. We calculate the cocycle condition of the affine twist cocycle on the case of σ .

$$\begin{aligned} \mathbf{AT}_h(\sigma) &= \sum_{i=1}^n \sigma_1^1 \cdots \sigma_{i-1}^2 \mathbf{AT}_h(\sigma_i^1) + \sum_{j=1}^n \sigma_1^1 \cdots \sigma_j^1 \mathbf{AT}_h(\sigma_j^2) \\ &= \sum_{j=1}^n \sigma_1^1 \cdots \sigma_j^1 \mathbf{AT}_h(\sigma_j^2). \end{aligned}$$

Next we calculate the Margulis invariant. Set $\mathbf{Y}^0 := \mathbf{Y}_h^0$.

$$\begin{aligned} \text{Mar}_{\mathbf{AT}_h}(\sigma) &= B(\mathbf{X}_\sigma^0, \sum_{j=1}^n \sigma_1^1 \cdots \sigma_j^1 \mathbf{AT}_h(\sigma_j^2)) \\ &= \sum_{j=1}^n B(\mathbf{X}_\sigma^0, \sigma_1^1 \cdots \sigma_j^1 \mathbf{AT}_h(\sigma_j^2)) \\ &= \sum_{j=1}^n B((\sigma_1^1 \cdots \sigma_j^1)^{-1} \mathbf{X}_\sigma^0, \mathbf{AT}_h(\sigma_j^2)) \cdots (\star) \end{aligned}$$

We note that $\phi \mathbf{X}_\sigma^0 = \mathbf{X}_{\phi\sigma\phi^{-1}}^0$ for every $\phi \in \text{SO}^o(2, 1)$. So we have

$$(\star) = \sum_{j=1}^n B(\mathbf{X}_{\sigma_j^2 \sigma_{j+1}^1 \sigma_{j+1}^2 \cdots \sigma_n^2 \sigma_1^1 \cdots \sigma_j^1}^0, \mathbf{AT}_h(\sigma_j^2)).$$

Then, by $\mathbf{AT}_h(\sigma_j^2) = \mathbf{Y}^0 - \sigma_j^2 \mathbf{Y}^0$, the Margulis invariant $\text{Mar}_{\mathbf{AT}_h}(\sigma)$ is equal to:

$$\sum_{j=1}^n B(\mathbf{X}_{\sigma_j^2 \sigma_{j+1}^1 \sigma_{j+1}^2 \dots \sigma_n^2 \sigma_1^1 \dots \sigma_j^1}^0, \mathbf{Y}^0) - \sum_{j=1}^n B(\mathbf{X}_{\sigma_{j+1}^1 \sigma_{j+1}^2 \dots \sigma_n^2 \sigma_1^1 \dots \sigma_j^1 \sigma_j^2}^0, \mathbf{Y}^0).$$

Consider the first terms $B(\mathbf{X}_{\sigma_j^2 \sigma_{j+1}^1 \sigma_{j+1}^2 \dots \sigma_n^2 \sigma_1^1 \dots \sigma_j^1}^0, \mathbf{Y}^0)$. We find these two unit vectors to satisfy Lemma 2.1, since, in \mathbb{H}^2 , $\sigma_j^2 \dots \sigma_j^1$ acts onto the a side of S_2 from S_1 . Therefore we can put as $\cos(\theta_h^\sigma)_{p_j}$ the value of this inner product.

We check the second terms. Notice that a loop $\sigma_{j+1}^1 \dots \sigma_j^2$ goes to S_1 from S_2 . So we can denote its intersection with h by q_j . The angle $(\theta_h^\sigma)_{q_j}$ at q_j satisfies

$$B(\mathbf{X}_{\sigma_{j+1}^1 \dots \sigma_j^2}^0, \mathbf{Y}^0) = \cos(\pi - (\theta_h^\sigma)_{q_j}) = -\cos(\theta_h^\sigma)_{q_j}.$$

So, $\text{Mar}_{\mathbf{AT}_h}(\sigma) = \sum_{j=1}^n \cos(\theta_h^\sigma)_{p_j} + \sum_{j=1}^n \cos(\theta_h^\sigma)_{q_j}$ holds. The proof of the other case for σ' is same. \square

Note that words which start from an element of Σ_2 also satisfy the equation (1). Thus we obtain the equation (1) for any loop in $\pi_1(S)$. \square

From Theorem 1.2 and the cosine formula by Wolpert[W], we can identify \mathbf{AT}_h with the tangent vector on the Teichmüller space of $S_{g,b}$ corresponding to a Fenchel-Nielsen twist along the separating loop h . In general, Fenchel-Nielsen twists are along some geodesic loops on $S_{g,b}$, which are pairwise disjoint.

Corollary 5.2. *For a linear sum of some affine twist cocycles whose associated simple closed curves are pairwise disjoint, the cosine formula in Theorem 1.2 holds if every associated curve is a separating curve.*

Namely, for a cocycle $\mathbf{at} := \sum_{k=1}^s \tau_k \mathbf{AT}_{h_k}$ and any geodesic loop σ in $S_{g,b}$,

$$(11) \quad \left. \frac{d\ell_\sigma}{dt}(t) \right|_{t=0} = 2 \sum_{k=1}^s \left\{ \tau_k \sum_{p_k \in h_k \cap \sigma} \cos(\theta_{h_k}^\sigma)_{p_k} \right\}$$

holds, where h_1, \dots, h_s are separating curves satisfying the assumption of this corollary.

Proof. This corollary is proved by linearity of the Lorentzian inner product B . \square

We can also identify the sum **at** of the affine twist cocycles with a tangent vector (on Teichmüller space) corresponding to the Fenchel-Nielsen twists along h_1, \dots, h_s .

REFERENCES

- [CD] V. Charette, T. Drumm, Strong marked isospectrality of affine Lorentzian groups, *J. Differential Geom.*, **66**(2004), *no.3*, 437–452
- [CDG1] V. Charette, T. Drumm, W. Goldman, Affine deformations of a three-holed sphere, *Geom. Topol.*, **14** (2010), *no.3*, 1352–1382.
- [CDG2] V. Charette, T. Drumm, W. Goldman, Finite-sided deformation spaces of complete affine 3-manifolds, *J. Topol.*, **7** (2014), *no.1*, 225 – 246.
- [CDG3] V. Charette, T. Drumm, W. Goldman, Proper affine deformation spaces of two-generator Fuchsian groups, arXiv:1501.04535v1[math.GT] (2015.1.19).
- [CG] V. Charette, W. Goldman, Mcshane-type identities for affine deformations, arXiv:1507.01854[math.GT] (2016.5.13),
- [DG1] T. Drumm, W. Goldman, Complete flat Lorentz 3-manifolds with free fundamental group, *Internat. J. Math.*, **1** (1990), *no.2*, 149–161.
- [DG2] T. Drumm, W. Goldman, Isospectrality of flat Lorentz 3-manifolds, *J. Differential Geom.*, **58** (2001), *no.3*, 457–465.
- [DGK1] J.Danciger, F.Gueritaud, F.Kassel, Geometry and topology of complete Lorentz spacetimes of constant curvature, arXiv:1306.2240[math.GT] (2013.6.10)
- [DGK2] J.Danciger, F.Gueritaud, F.Kassel, Margulis spacetimes via the arc complex, *Invent.Math.*, **204**(2016), *no.1*, 133 – 193.
- [DGK3] J.Danciger, F.Gueritaud, F.Kassel, Fundamental domains for free groups acting on anti-de Sitter 3-space, arXiv:1410.5804[math.GT] (2014.10.21).
- [G] W. Goldman, The Margulis invariant of isometric actions on Minkowski $(2 + 1)$ -space, *Springer, Berlin* (2002), 149-164.
- [GM] W. Goldman, G. Margulis, Flat Lorentz 3-manifolds and cocompact Fuchsian groups, *Contemp. Math.*, **262** (2000)
- [M] T. Masuda, Affine twist deformation of a sphere with holes, *Geom.Dedicata*, **182** (2016), 249–262.
- [Ma] G. Margulis, Free completely discontinuous groups of affine transformations, *Soviet Math.Dokl.*, **28** (1983), *no.2*, 435–439.
- [W] S. Wolpert, An elementary formula for the Fenchel-Nielsen twist, *Comment.Math.Helv.*, **56** (1981), *no.1*, 132–135

(Masuda) DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE,
 OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN
E-mail address: t-masuda@cr.math.sci.osaka-u.ac.jp